

Existence and uniqueness of solution for the stochastic nonlinear diffusion equation of plasma

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Abstract

In this paper we are concerned with the stochastic partial differential equations of superfast diffusion processes describing behavior of plasma $dX(t) - \Delta \operatorname{sign}(X(t)) \ln(|X(t)| + 1) dt = \sqrt{\mathcal{Q}} dW(t)$, in $(0, T) \times \mathcal{O}$, where \mathcal{O} is a bounded open interval of \mathbb{R} . We define a strong solution adequate to the properties of the natural logarithm and we prove the corresponding existence and uniqueness result.

Key word: stochastic PDE's, monotone operators, super-fast diffusion, plasma physics

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Introduction

Consider a nonlinear diffusion process of the following form

$$dX(t) = \Delta \ln(X(t) + 1) dt \quad (1)$$

where $X(t, \xi)$ is the density for the time - space coordinates (t, ξ) . This equation describes the process that has been observed during experiments using Wisconsin toroidal octupole plasma containment device (see [16]). Kamimura and Dawson predicted in [17] this process for cross-field conservative diffusion of plasma including mirror effects.

The same equation describes the expansion of a thermalized electron cloud and arises also in studies of the central limit approximation to Carleman's model of the Boltzmann equation (see [12] and [18]). The asymptotic behavior of this equation was studied in [9]. Most of the natural phenomena exhibit variability which cannot be modeled by using deterministic approaches. More accurately,

natural systems can be represented as stochastic models and the deterministic description can be considered as the subset of the pertinent stochastic models.

The purpose of this paper is to analyze such equations within the framework of stochastic evolution equations with (1) as underlying motivating example. Let us now introduce the suitable framework for this problem.

Notation

Let \mathcal{O} be a bounded open interval of \mathbb{R} . Recall the distribution spaces $H_0^1(\mathcal{O})$ on \mathcal{O} and it's dual $H^{-1}(\mathcal{O})$ with the scalar product and the norm given by

$$\langle u, v \rangle_{-1} = \left((-\Delta)^{-1} u, v \right)_2, \quad |u|_{-1} = \left((-\Delta)^{-1} u, u \right)_2^{1/2},$$

respectively, where $(\cdot, \cdot)_2$ is the pairing between $H_0^1(\mathcal{O})$ and $H^{-1}(\mathcal{O})$ and the scalar product of $L^2(\mathcal{O})$. The norm in $L^p(\mathcal{O})$, $1 \leq p \leq \infty$ will be denoted by $|\cdot|_p$.

Given a Hilbert space U , the norm of U will be denoted by $|\cdot|_U$ and the scalar product by $(\cdot, \cdot)_U$. By $C([0, T]; U)$ we shall denote the space of U -valued continuous functions on $[0, T]$ and by $C_W([0, T]; L^2(\Omega, \mathcal{F}, \mathbb{P}; U))$ the space of all U -valued adapted stochastic processes with respect to filtration \mathcal{F} of the probability space, which are mean square continuous.

Formulation of the problem and hypotheses

The main result is an existence and uniqueness theorem for the following stochastic nonlinear diffusion equations in $H^{-1}(\mathcal{O})$ with additive noise

$$\begin{cases} dX(t) - \Delta \operatorname{sign}(X(t)) \ln(|X(t)| + 1) dt = \sqrt{Q} dW(t), & \text{in } (0, T) \times \mathcal{O}, \\ \operatorname{sign}(X(t)) \ln(|X(t)| + 1) = 0, & \text{on } (0, T) \times \partial\mathcal{O}, \\ X(0) = x, & \text{in } \mathcal{O}, \end{cases} \quad (2)$$

where \mathcal{O} is an open bounded interval of \mathbb{R} , x is an initial datum and

$$\operatorname{sign}(x) = \begin{cases} \frac{x}{|x|}, & \text{if } x \neq 0; \\ [-1, 1], & \text{if } x = 0. \end{cases}$$

Here $W(t)$ is a cylindrical Wiener process on $L^2(\mathcal{O})$ of the form

$$W(t) = \sum_{k=1}^{\infty} \beta_k(t) e_k, \quad t \geq 0$$

for $\{\beta_k\}$ a sequence of independent standard Brownian motion on a filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P})$ and $\{e_k\}$ is a complete orthonormal system in $L^2(\mathcal{O})$ of eigenfunctions of $-\Delta$ with Dirichlet homogeneous boundary conditions. We denote by $\{\lambda_k\}$ the corresponding sequence of eigenvalues. The operator $Q \in L(L^2(\mathcal{O}))$ defined by

$$\sqrt{Q}h = \sum_{k=1}^{\infty} \gamma_k \langle h, e_k \rangle_2 e_k, \quad \forall h \in L^2(\mathcal{O})$$

where $\{\gamma_k\}$ is a sequence of positive numbers, is symmetric, self-adjoint and nonnegative. Then the random forcing term is

$$\sqrt{Q}dW(t) = \sum_{k=1}^{\infty} \gamma_k e_k d\beta_k(t), \quad t \geq 0.$$

Because $\Psi : \mathbb{R} \rightarrow \mathbb{R}$, $\Psi(x) = \text{sign}(x) \ln(|x| + 1)$ is a maximal monotone operator, we can see that the operator defined by $A(x) = -\Delta \Psi(x)$, for all $x \in D(A)$, where $D(A) = \{x \in H^{-1}(\mathcal{O}) \cap L^1(\mathcal{O}); \Psi(x) \in H_0^1(\mathcal{O})\}$ is maximal monotone in $H^{-1}(\mathcal{O}) \times H^{-1}(\mathcal{O})$ (see [11]).

In this paper we shall assume that the sequence $\{\gamma_k\}$ is such that

$$(H_1) \quad \sum_{k=1}^{\infty} \gamma_k^2 \lambda_k^2 < \infty \text{ and } \sum_{k=1}^{\infty} \gamma_k \lambda_k^3 < \infty;$$

and

$$(H_2) \quad \sqrt{Q}W \in C([0, T] \times \overline{\mathcal{O}}) \quad \mathbb{P} - a.s..$$

Note that, for every ω fixed, we have, for some constant C , that

$$\sup_{s \in [0, T]} \left| \sqrt{Q}W(s) \right|_{L^\infty(\mathcal{O})} \leq C. \quad (3)$$

A similar result was proven in [2] for semilinear parabolic stochastic equations and in [5] for porous media stochastic equations.

Denote by $g : \mathbb{R} \rightarrow \mathbb{R}$,

$$g(x) = (|x| + 1) \ln(|x| + 1) - |x|,$$

and note that, in our case, $\partial g = \Psi$.

Definition 1 *An adapted stochastic process*

$$X \in C_W([0, T]; H^{-1}(\mathcal{O})) \cap L^p((0, T) \times \mathcal{O} \times \Omega), \quad p \geq 4,$$

is said to be a solution to equation (2) if

$$\begin{aligned} & \frac{1}{2} \left| X(t) - \sqrt{Q}W(t) - Z(t) \right|_{-1}^2 + \int_0^t \int_{\mathcal{O}} g(X(s)) d\xi ds \\ & + \int_0^t \int_{\mathcal{O}} (-\Delta)^{-1} \frac{\partial}{\partial s} Z(s) \left(X(s) - \sqrt{Q}W(s) - Z(s) \right) d\xi ds \\ & \leq \frac{1}{2} |x - Z(0)|_{-1}^2 + \int_0^t \int_{\mathcal{O}} g(Z(s) + \sqrt{Q}W(s)) d\xi ds, \quad \mathbb{P} - a.s. \end{aligned}$$

for every starting point $x \in L^p(\mathcal{O})$ and for all adapted stochastic processes

$$Z \in C_W([0, T]; H^{-1}(\mathcal{O})),$$

such that, for every $\omega \in \Omega$ fixed, satisfies

- i) $Z \in C([0, T]; L^2(\mathcal{O}))$;
- ii) $Z' \in L^2(0, T; H^{-1}(\mathcal{O}))$;
- iii) $g(Z + \sqrt{Q}W) \in L^1((0, T) \times \mathcal{O})$.

Definition 1 resembles the classical definition of a mild (integral) solution to deterministic variational inequality (see, e.g., [1]). For stochastic differential equations a slightly different version was used in [6] and [23]. We can easily see that a solution in the sense of [[20], Definition 4.2.1] is also a solution in the sense of Definition 1 above.

Context

Existence results for equation

$$\begin{cases} dX(t) - \Delta \Psi(X(t)) dt = \sqrt{Q} dW(t), & \text{in } (0, T) \times \mathcal{O}, \\ \Psi(X(t)) = 0, & \text{on } (0, T) \times \partial\mathcal{O}, \\ X(0) = x, & \text{in } \mathcal{O}, \end{cases}$$

were obtained in [4] for Ψ monotonically increasing, continuous, with $\Psi(0) = 0$, and satisfying the following growth conditions

$$\Psi'(r) \leq \alpha_1 |r|^{m-1} + \alpha_2 \text{ and } \int_0^r \Psi(s) ds \geq \alpha_3 |r|^{m+1} + \alpha_4,$$

for all $r \in \mathbb{R}$, where $\alpha_1, \alpha_2, \alpha_4 \geq 0$, $\alpha_3 > 0$ and $m \geq 1$. This result was generalized in [7]. Note that our case is not covered by those hypotheses.

An other existence result was proved in [22] for the operator

$$\Psi(r) = \text{sign}(r) |r|^{\theta-1} (\log(|r| + 1))^s,$$

$r \in \mathbb{R}$ and $\theta \in (1, \infty)$, $s \in [1, \infty)$ (see [22] Example 3.5).

In the present paper we are considering the critical case $\theta = 1$ which was not covered, by using a different approach and a different definition of the solution.

1 The main result

The main result of this work is the following

Theorem 2 *For all $x \in L^p(\mathcal{O})$, $p \geq 4$, equation (2) has an unique solution in the sense of Definition 1.*

In order to prove this result we need some estimates that will be used for both existence and uniqueness.

A priori Estimates

Denote by $\Psi : \mathbb{R} \rightarrow 2^{\mathbb{R}}$, $\Psi(x) = \text{sign}(x) \ln(|x| + 1)$. Since Ψ is a maximal monotone operator we can consider the following approximating equation

$$\begin{cases} dX_\varepsilon(t) - \Delta \bar{\Psi}_\varepsilon(X_\varepsilon(t)) dt = \sqrt{Q} dW(t), & \text{in } (0, T) \times \mathcal{O}, \\ \bar{\Psi}_\varepsilon(X_\varepsilon(t)) = 0, & \text{on } (0, T) \times \partial\mathcal{O}, \\ X_\varepsilon(0) = x, & \text{in } \mathcal{O}, \end{cases} \quad (4)$$

where $\bar{\Psi}_\varepsilon(x) = \Psi_\varepsilon(x) + \varepsilon x$, for all $x \in \mathbb{R}$, and Ψ_ε is the Yosida approximation of Ψ , i.e.,

$$\Psi_\varepsilon(x) = \frac{1}{\varepsilon} \left(1 - (1 + \varepsilon \Psi)^{-1} \right)(x) = \Psi \left((1 + \varepsilon \Psi)^{-1} x \right)$$

for all $\varepsilon > 0$. We can take this approximation since Ψ is a maximal monotone operator (see e.g., [1], [3]).

For each $\varepsilon > 0$ fixed, equation (4) has an unique solution in the sense of [15] or [[20], Definition 4.2.1] (see Example 4.1.11 from [20]). Note that solution X_ε to the approximation equation (4) is in our case a path-wise continuous, $H^{-1}(\mathcal{O})$ -valued, (\mathcal{F}_t) -adapted stochastic process. Clearly, this is also solution in the sense of Definition 1.

Setting

$$Y_\varepsilon(t) = X_\varepsilon(t) - \sqrt{Q}W(t),$$

we may rewrite (4) as a random equation

$$\begin{cases} dY_\varepsilon(t) - \Delta \bar{\Psi}_\varepsilon(Y_\varepsilon(t) + \sqrt{Q}W(t)) dt = 0, & \text{in } (0, T) \times \mathcal{O}, \\ \bar{\Psi}_\varepsilon(Y_\varepsilon(t) + \sqrt{Q}W(t)) = 0, & \text{on } (0, T) \times \partial\mathcal{O}, \\ Y_\varepsilon(0) = x, & \text{in } \mathcal{O}, \end{cases} \quad (5)$$

For each $\omega \in \Omega$ fixed, by classical existence theory for nonlinear equation we have that equation (5) has a unique solution $Y_\varepsilon \in C([0, T]; L^2(\mathcal{O}))$, with $Y'_\varepsilon \in L^2(0, T; H^{-1}(\mathcal{O}))$ (see [19] for the general result and [5] for a similar case).

By the Itô formula with the function $x \mapsto |x|_2^2$ we get from (4) that

$$\mathbb{E} |X_\varepsilon(t, x)|_2^2 \leq |x|_2^2 + t \sum_{k=1}^{\infty} \lambda_k^2 \gamma_k^2$$

and then, we get that, for each $\omega \in \Omega$ fixed, we have

$$|X_\varepsilon(t)|_{L^2(\mathcal{O})}^2 \leq C(\omega), \quad (6)$$

for all $t \in [0, T]$, with C independent of ε . By assumption **H**₂, we can assume the same estimate holds for Y_ε , i.e.,

$$|Y_\varepsilon(t)|_{L^2(\mathcal{O})}^2 = |X_\varepsilon(t) - \sqrt{Q}W(t)|_{L^2(\mathcal{O})}^2 \leq C, \quad (7)$$

for all $t \in [0, T]$, with C independent of ε .

Lemma 3 *There exists a constant C (independent of ε) such that for all $\omega \in \Omega$ fixed, we have that*

$$\int_0^T \int_{\mathcal{O}} \left| \Psi_{\varepsilon} \left(Y_{\varepsilon}(s) + \sqrt{Q}W(s) \right) \right| d\xi ds \leq C,$$

for all $t \in [0, T]$.

Proof of Lemma 3.

Recall that $g : \mathbb{R} \rightarrow \mathbb{R}$, is defined by

$$g(x) = (|x| + 1) \ln(|x| + 1) - |x|$$

and g_{ε} is the Moreau-Yosida approximation of g . Since $\partial g = \Psi$ we have by [[1], Theorem 2.2] that $\nabla g_{\varepsilon} = \Psi_{\varepsilon}$.

From the definition of the subdifferential we have, for all $0 < \lambda < 1$ fixed and for all θ , such that $|\theta| < \frac{\lambda}{2}$, the following inequality

$$\begin{aligned} & \Psi_{\varepsilon} \left(Y_{\varepsilon} + \sqrt{Q}W \right) \left(Y_{\varepsilon} + \sqrt{Q}W - \theta - \lambda \right) \\ & \geq g_{\varepsilon} \left(Y_{\varepsilon} + \sqrt{Q}W \right) - g_{\varepsilon}(\theta + \lambda) \\ & \geq g \left(J_{\varepsilon} \left(Y_{\varepsilon} + \sqrt{Q}W \right) \right) - g(\theta + \lambda), \end{aligned}$$

a.e. on $[0, T] \times \mathcal{O}$.

Taking into account that

$$0 \leq g(x) \leq x^2, \text{ for all } x \in \mathbb{R},$$

we obtain that

$$\begin{aligned} \Psi_{\varepsilon} \left(Y_{\varepsilon} + \sqrt{Q}W \right) \left(Y_{\varepsilon} + \sqrt{Q}W - \theta - \lambda \right) & \geq -(\theta + \lambda)^2 \\ & > -\frac{9\lambda^2}{4} > -\frac{9}{4}, \end{aligned}$$

a.e. on $[0, T] \times \mathcal{O}$.

By taking

$$\theta = \frac{\lambda}{2} \frac{\Psi_{\varepsilon} \left(Y_{\varepsilon} + \sqrt{Q}W \right)}{\left| \Psi_{\varepsilon} \left(Y_{\varepsilon} + \sqrt{Q}W \right) \right|}$$

it follows that

$$\left| \Psi_{\varepsilon} \left(Y_{\varepsilon} + \sqrt{Q}W \right) \right| \leq \frac{2}{\lambda} \Psi_{\varepsilon} \left(Y_{\varepsilon} + \sqrt{Q}W \right) \left(Y_{\varepsilon} + \sqrt{Q}W - \lambda \right) + \frac{9}{2\lambda}$$

a.e. on $[0, T] \times \mathcal{O}$.

Consequently we obtain that

$$\begin{aligned}
& \int_0^T \int_{\mathcal{O}} \left| \Psi_{\varepsilon} \left(Y_{\varepsilon}(s) + \sqrt{Q}W \right) (s) \right| d\xi ds \quad (8) \\
& \leq \frac{2}{\lambda} \int_0^T \int_{\mathcal{O}} \Psi_{\varepsilon} \left(Y_{\varepsilon} + \sqrt{Q}W \right) \left(Y_{\varepsilon} + \sqrt{Q}W - \lambda \right) d\xi ds + \frac{9}{2\lambda} |\mathcal{O}| T \\
& \leq \frac{2}{\lambda} \int_0^T \int_{\mathcal{O}} \left[(-\Delta)^{-1} \left(-\frac{\partial}{\partial s} Y_{\varepsilon} \right) - \varepsilon \left(Y_{\varepsilon} + \sqrt{Q}W \right) \right] \\
& \quad \times \left(Y_{\varepsilon} + \sqrt{Q}W - \lambda \right) d\xi ds + \frac{9}{2\lambda} |\mathcal{O}| T \\
& \leq \frac{2}{\lambda} \int_0^T \int_{\mathcal{O}} \left[(-\Delta)^{-1} \left(-\frac{\partial}{\partial s} Y_{\varepsilon} \right) \right] \left(Y_{\varepsilon} + \sqrt{Q}W - \lambda \right) d\xi ds \\
& \quad - \frac{2\varepsilon}{\lambda} \int_0^T \int_{\mathcal{O}} \left(Y_{\varepsilon} + \sqrt{Q}W \right)^2 d\xi ds \\
& \quad + 2\varepsilon \int_0^T \int_{\mathcal{O}} \left(Y_{\varepsilon} + \sqrt{Q}W \right) d\xi ds + \frac{9}{2\lambda} |\mathcal{O}| T \\
& \leq \frac{2}{\lambda} \int_0^T \int_{\mathcal{O}} \left[(-\Delta)^{-1} \left(-\frac{\partial}{\partial s} Y_{\varepsilon} \right) \right] \left(Y_{\varepsilon} + \sqrt{Q}W - \lambda \right) d\xi ds + C.
\end{aligned}$$

Now it is sufficient to show boundedness for

$$\begin{aligned}
& \frac{2}{\lambda} \int_0^T \int_{\mathcal{O}} (-\Delta)^{-1} \left(-\frac{\partial}{\partial s} Y_{\varepsilon} \right) \left(Y_{\varepsilon} + \sqrt{Q}W - \lambda \right) d\xi ds \quad (9) \\
& = \frac{2}{\lambda} \int_0^T \int_{\mathcal{O}} (-\Delta)^{-1} \left(-\frac{\partial}{\partial s} Y_{\varepsilon} \right) Y_{\varepsilon} d\xi ds \\
& \quad + \frac{2}{\lambda} \int_0^T \int_{\mathcal{O}} (-\Delta)^{-1} \left(-\frac{\partial}{\partial s} Y_{\varepsilon} \right) \sqrt{Q}W d\xi ds \\
& \quad - 2 \int_0^T \int_{\mathcal{O}} (-\Delta)^{-1} \left(-\frac{\partial}{\partial s} Y_{\varepsilon} \right) d\xi ds \stackrel{\text{denote}}{=} I_1 + I_2 + I_3.
\end{aligned}$$

Firstly, we have that

$$\begin{aligned}
I_1 &= \frac{2}{\lambda} \int_0^T \int_{\mathcal{O}} (-\Delta)^{-1} \left(-\frac{\partial}{\partial s} Y_{\varepsilon} \right) Y_{\varepsilon} d\xi ds = \int_0^T \left\langle -\frac{\partial}{\partial s} Y_{\varepsilon}, Y_{\varepsilon} \right\rangle_{-1} ds \\
&= -\frac{1}{2} \left(|Y_{\varepsilon}(T)|_{-1}^2 - |Y_{\varepsilon}(0)|_{-1}^2 \right) \leq C.
\end{aligned}$$

For the second term

$$I_2 = \frac{2}{\lambda} \int_0^T \int_{\mathcal{O}} (-\Delta)^{-1} \left(-\frac{\partial}{\partial s} Y_{\varepsilon} \right) \sqrt{Q}W d\xi ds,$$

we may choose, for $\alpha > 0$, small enough, a decomposition $0 = t_0 < t_1 < \dots < t_i < t_{i+1} < \dots < t_N = T$ such that, for all $t, s \in [t_i, t_{i+1}]$, we have

$$\left| \sqrt{Q}W(t) - \sqrt{Q}W(s) \right|_{L^{\infty}(\mathcal{O})} < \alpha.$$

Consequently, we may write

$$\begin{aligned}
I_2 &= \frac{2}{\lambda} \sum_{i=0}^{N-1} \int_{t_i}^{t_{i+1}} \int_{\mathcal{O}} (-\Delta)^{-1} \left(-\frac{\partial}{\partial s} Y_\varepsilon \right) \sqrt{Q} W d\xi ds \\
&= \frac{2}{\lambda} \sum_{i=0}^{N-1} \int_{t_i}^{t_{i+1}} \int_{\mathcal{O}} (-\Delta)^{-1} \left(-\frac{\partial}{\partial s} Y_\varepsilon \right) \left(\sqrt{Q} W(s) - \sqrt{Q} W(t_i) \right) d\xi ds \\
&\quad + \frac{2}{\lambda} \sum_{i=0}^{N-1} \int_{t_i}^{t_{i+1}} \int_{\mathcal{O}} (-\Delta)^{-1} \left(-\frac{\partial}{\partial s} Y_\varepsilon \right) \sqrt{Q} W(t_i) d\xi ds \\
&\leq \frac{2\alpha}{\lambda} \int_0^T \left| \Psi_\varepsilon \left(Y_\varepsilon(s) + \sqrt{Q} W(s) \right) \right|_{L^1(\mathcal{O})} ds \\
&\quad - \frac{2}{\lambda} \sum_{i=0}^{N-1} \int_{t_i}^{t_{i+1}} \left\langle \frac{\partial}{\partial s} Y_\varepsilon(s), \sqrt{Q} W(t_i) \right\rangle_{-1} ds \\
&\leq \frac{2\alpha}{\lambda} \int_0^T \left| \Psi_\varepsilon \left(Y_\varepsilon(s) + \sqrt{Q} W(s) \right) \right|_{L^1(\mathcal{O})} ds + C.
\end{aligned}$$

Finally we have

$$\begin{aligned}
I_3 &= 2 \int_0^T \int_{\mathcal{O}} (-\Delta)^{-1} \left(\frac{\partial}{\partial s} Y_\varepsilon \right) d\xi ds \\
&= 2 \int_{\mathcal{O}} \left((-\Delta)^{-1} Y_\varepsilon(T) - (-\Delta)^{-1} Y_\varepsilon(0) \right) d\xi \\
&\leq 2 \left(\left| (-\Delta)^{-1} Y_\varepsilon(T) \right|_{L^1(\mathcal{O})} + \left| (-\Delta)^{-1} Y_\varepsilon(0) \right|_{L^1(\mathcal{O})} \right) \leq C,
\end{aligned}$$

since $H_0^1(\mathcal{O}) \subset L^1(\mathcal{O})$ and $\left| (-\Delta)^{-1} x \right|_{H_0^1(\mathcal{O})} = |x|_{-1}$, for all $x \in H^{-1}(\mathcal{O})$.

Going back to (9) we get that

$$\begin{aligned}
&\frac{2}{\lambda} \int_0^T \int_{\mathcal{O}} (-\Delta)^{-1} \left(-\frac{\partial}{\partial s} Y_\varepsilon \right) \left(Y_\varepsilon + \sqrt{Q} W - \lambda \right) d\xi ds \\
&\leq \frac{2\alpha}{\lambda} \int_0^T \left| \Psi_\varepsilon \left(Y_\varepsilon(s) + \sqrt{Q} W(s) \right) \right|_{L^1(\mathcal{O})} ds + C
\end{aligned}$$

and then, for α small enough and λ fixed, we get from (8) that

$$\int_0^T \int_{\mathcal{O}} \left| \Psi_\varepsilon \left(Y_\varepsilon(s) + \sqrt{Q} W(s) \right) \right| d\xi ds \leq C.$$

The proof of the lemma is now complete. ■

Proof of the main result.

Existence

We have to prove existence of the limit for $\{Y_\varepsilon\}_\varepsilon$ as $\varepsilon \rightarrow 0$ and consequently we'll get existence of the solution for equation (2) in the sense of Definition 1. From Lemma 3 we get, for all $\omega \in \Omega$ fixed, that

$$\left| \frac{\partial}{\partial t} (-\Delta)^{-1} Y_\varepsilon \right|_{L^1(0,T;L^1(\mathcal{O}))} \leq C. \quad (10)$$

Since for $\mathcal{O} \subset \mathbb{R}$ we have $L^1(\mathcal{O}) \subset H^{-1}(\mathcal{O})$, this leads to

$$\begin{aligned} V_0^T \left((-\Delta)^{-1} Y_\varepsilon \right) &\leq \left| \frac{\partial}{\partial t} (-\Delta)^{-1} Y_\varepsilon \right|_{L^1(0,T;H^{-1}(\mathcal{O}))} \\ &\leq \left| \frac{\partial}{\partial t} (-\Delta)^{-1} Y_\varepsilon \right|_{L^1(0,T;L^1(\mathcal{O}))} \leq C, \end{aligned} \quad (11)$$

for C independent of ε . We denoted by

$$V_0^T(f) = \sup \sum_{i=1}^n |f(t_i) - f(t_{i-1})|_{-1}$$

where the supremum is taken over all partitions $D = \{0 = t_0 < t_1 < \dots < t_n = T\}$ of $[0, T]$.

On the other hand, by classical deterministical arguments we have that, for each $\omega \in \Omega$ fixed

$$\sup_{t \in [0, T]} |Y_\varepsilon(t)|_{-1}^2 \leq C(\omega)$$

which leads to

$$\sup_{t \in [0, T]} \left| (-\Delta)^{-1} Y_\varepsilon(t) \right|_{H_0^1(\mathcal{O})}^2 \leq C(\omega). \quad (12)$$

Then, since $\left\{ (-\Delta)^{-1} Y_\varepsilon(t) \right\}_\varepsilon$ is bounded in $H_0^1(\mathcal{O})$ and $H_0^1(\mathcal{O}) \subset H^{-1}(\mathcal{O})$ compactly, we have that

$$\left\{ (-\Delta)^{-1} Y_\varepsilon(t) \right\}_\varepsilon \text{ is compact in } H^{-1}(\mathcal{O}), \text{ for all } t \in [0, T]. \quad (13)$$

From (11) and (13) it follows, via Helly-Foias theorem (see Theorem 3.5 and Remark 3.2 from [8] or page 238 from [21]), that, on a subsequence, we have

$$(-\Delta)^{-1} Y_\varepsilon(t) \rightarrow G(t) \text{ strongly in } H^{-1}(\mathcal{O}), \text{ for all } t \in [0, T].$$

Because $H_0^1(\mathcal{O}) \subset L^2(\mathcal{O}) \subset H^{-1}(\mathcal{O})$ we have that

$$\begin{aligned} \left| (-\Delta)^{-1} Y_\varepsilon(t) - G(t) \right|_{L^2(\mathcal{O})} &\leq \varepsilon \left| (-\Delta)^{-1} Y_\varepsilon(t) - G(t) \right|_{H_0^1(\mathcal{O})} \\ &\quad + C(\varepsilon) \left| (-\Delta)^{-1} Y_\varepsilon(t) - G(t) \right|_{-1} \end{aligned}$$

for all $t \in [0, T]$ (see [19], p. 58) and therefore

$$(-\Delta)^{-1} Y_\varepsilon(t) \rightarrow G(t) \text{ strongly in } L^2(\mathcal{O}), \text{ for all } t \in [0, T].$$

Using (7) we obtain that

$$(-\Delta)^{-1} Y_\varepsilon(t) \rightarrow (-\Delta)^{-1} Y(t) \text{ strongly in } L^2(\mathcal{O}), \text{ for all } t \in [0, T]. \quad (14)$$

On the other hand we have

$$\begin{aligned} & |Y_\varepsilon(t) - Y_\lambda(t)|_{-1}^2 \\ &= \left\langle (-\Delta)^{-1} Y_\varepsilon(t) - (-\Delta)^{-1} Y_\lambda(t), Y_\varepsilon(t) - Y_\lambda(t) \right\rangle_{L^2(\mathcal{O})} \\ &\leq \left| (-\Delta)^{-1} Y_\varepsilon(t) - (-\Delta)^{-1} Y_\lambda(t) \right|_{L^2(\mathcal{O})} |Y_\varepsilon(t) - Y_\lambda(t)|_{L^2(\mathcal{O})}. \end{aligned}$$

Using (7) and (14) we get that

$$Y_\varepsilon(t) \rightarrow Y(t) \text{ strongly in } H^{-1}(\mathcal{O}), \text{ for all } t \in [0, T] \quad (15)$$

and, since $X_\varepsilon = Y_\varepsilon + \sqrt{Q}W$, with $\sqrt{Q}W \in C([0, T] \times \overline{\mathcal{O}})$, we have

$$X_\varepsilon(t) \rightarrow X(t) \text{ strongly in } H^{-1}(\mathcal{O}), \text{ for all } t \in [0, T]. \quad (16)$$

On the other hand, from (6) we have, for every $\omega \in \Omega$ fixed, that

$$\begin{aligned} \int_0^t \int_{\mathcal{O}} g\left((1 + \varepsilon\Psi)^{-1} X_\varepsilon(t)\right) d\xi &\leq \int_0^t \int_{\mathcal{O}} \left|(1 + \varepsilon\Psi)^{-1} X_\varepsilon(t)\right|^2 d\xi \\ &\leq \int_0^t \int_{\mathcal{O}} |X_\varepsilon(t)|^2 d\xi \leq C(\omega), \text{ for all } t \in [0, T], \end{aligned}$$

where $(1 + \varepsilon\Psi)^{-1}$ is the resolvent of Ψ .

Since

$$\lim_{|x| \rightarrow \infty} \frac{g(x)}{|x|} = \infty$$

we have that $\left\{ (1 + \varepsilon\Psi)^{-1} (X_\varepsilon(t)) \right\}_\varepsilon$ is bounded and equi-integrable in $L^1(\mathcal{O} \times (0, T))$.

Then, by the Dunford-Pettis theorem, we get that the sequence is weakly compact in $L^1(\mathcal{O} \times (0, T))$. Hence, along a subsequence, again denoting by ε , we obtain that

$$(1 + \varepsilon\Psi)^{-1} (X_\varepsilon(t)) \rightharpoonup X(t), \text{ weakly in } L^1(\mathcal{O} \times (0, T)), \quad (17)$$

as $\varepsilon \rightarrow 0$.

We know that $X_\varepsilon = Y_\varepsilon + \sqrt{Q}W$ is also a solution to equation (4) in the sense of our definition, *i.e.*,

$$\begin{aligned} & \frac{1}{2} \left| X_\varepsilon(t) - \sqrt{Q}W(t) - Z(t) \right|_{-1}^2 + \int_0^t \int_{\mathcal{O}} g_\varepsilon(X_\varepsilon(s)) d\xi ds \\ & \quad + \int_0^t \int_{\mathcal{O}} (-\Delta)^{-1} \frac{\partial}{\partial s} Z(s) \left(X_\varepsilon(s) - \sqrt{Q}W(s) - Z(s) \right) d\xi ds \\ & \leq \frac{1}{2} |x - Z(0)|_{H^{-1}(\mathcal{O})}^2 + \int_0^t \int_{\mathcal{O}} g_\varepsilon \left(Z(s) + \sqrt{Q}W(s) \right) d\xi ds, \quad \mathbb{P}\text{-a.s.} \end{aligned} \quad (18)$$

for all $t \in [0, T]$.

We intend to take the *liminf* for $\varepsilon \rightarrow 0$ in (18).

Convergence of the first term is a direct consequence of (16) and for the last term we only need to use classical properties of the Moreau-Yosida approximation i.e., $g \left((1 + \varepsilon \Psi)^{-1} (x) \right) \leq g_\varepsilon (x) \leq g (x)$ for all $x \in \mathbb{R}$.

We shall discuss now the second and the third term of the left hand side.

Since $\varphi : L^1(\mathcal{O} \times (0, T)) \rightarrow \mathbb{R}$ defined by $\varphi(x) = \int_0^t \int_{\mathcal{O}} g(x(\xi, s)) d\xi ds$ is weakly *l.s.c.* on $L^1(\mathcal{O} \times (0, T))$ (see [1] Proposition 2.9 and Proposition 2.12 from Chapter 2) we get from (17) that

$$\liminf_{\varepsilon \rightarrow 0} \int_0^t \int_{\mathcal{O}} g_\varepsilon(X_\varepsilon(s)) d\xi ds \geq \int_0^t \int_{\mathcal{O}} g(X(s)) d\xi ds, \quad \text{for all } t \in [0, T],$$

and then we can pass to the *liminf* for $\varepsilon \rightarrow 0$ in the second term.

The third term of the left hand side can be written as

$$\int_0^t \left\langle \frac{\partial}{\partial s} Z(s), Y_\varepsilon(s) - Z(s) \right\rangle_{-1} ds.$$

From (15) we have that

$$\left\langle \frac{\partial}{\partial s} Z, Y_\varepsilon - Z \right\rangle_{-1} \rightarrow \left\langle \frac{\partial}{\partial s} Z, Y - Z \right\rangle_{-1}, \quad \text{a.e. on } [0, T].$$

On the other hand we can easily see that

$$\begin{aligned} \left\langle \frac{\partial}{\partial s} Z(s), Y_\varepsilon(s) - Z(s) \right\rangle_{-1} &\leq \left| \frac{\partial}{\partial s} Z(s) \right|_{-1} |Y_\varepsilon(s) - Z(s)|_{-1} \\ &\leq \left| \frac{\partial}{\partial s} Z(s) \right|_{-1} \operatorname{ess\,sup}_{s \in [0, T]} |Y_\varepsilon(s) - Z(s)|_{-1} \\ &\leq C \left| \frac{\partial}{\partial s} Z(s) \right|_{-1} \in L^2(0, T), \quad \text{a.e. on } [0, T]. \end{aligned}$$

Now, by using Lebesgue's dominated convergence theorem we obtain that

$$\lim_{\varepsilon \rightarrow 0} \int_0^t \left\langle \frac{\partial}{\partial s} Z(s), Y_\varepsilon(s) - Z(s) \right\rangle_{-1} ds = \int_0^t \left\langle \frac{\partial}{\partial s} Z(s), Y(s) - Z(s) \right\rangle_{-1} ds.$$

At this point we can take the *liminf* for $\varepsilon \rightarrow 0$ in (18) and get that

$$\begin{aligned} &\frac{1}{2} \left| X(t) - \sqrt{Q}W(t) - Z(t) \right|_{-1}^2 + \int_0^t \int_{\mathcal{O}} g(X(s)) d\xi ds \\ &\quad + \int_0^t \int_{\mathcal{O}} (-\Delta)^{-1} \frac{\partial}{\partial s} Z(s) \left(X(s) - \sqrt{Q}W(s) - Z(s) \right) d\xi ds \\ &\leq \frac{1}{2} |x - Z(0)|_{-1}^2 + \int_0^t \int_{\mathcal{O}} g(Z(s) + \sqrt{Q}W(s)) d\xi ds, \quad \mathbb{P}\text{-a.s.} \end{aligned}$$

for all $t \in [0, T]$.

By applying the Itô formula to equation (4) with the function $x \mapsto |x|_{-1}^2$ and from (16) we get that

$$X_\varepsilon \rightarrow X, \text{ weakly in } L_W^2([0, T]; L^2(\Omega; H^{-1}(\mathcal{O}))), \text{ as } \varepsilon \rightarrow 0.$$

Then, arguing as in [20], we may replace X by a $H^{-1}(\mathcal{O})$ -continuous version and follows that the solution is also an (\mathcal{F}_t) -adapted stochastic process.

On the other hand we can easily see that $X \in L^p(\Omega \times \mathcal{O} \times [0, T])$ arguing as in Lemma 3.1 from [7] and that conclude the proof of the existence.

Uniqueness

Consider \bar{X} an arbitrary solution to equation (2) in the sense of Definition 1.

The main idea of the proof is to take $Z = (1 - \mu\Delta)^{-1} Y_\varepsilon = J_\mu Y_\varepsilon$ in Definition 1, for Y_ε the solution to equation (19) below and J_μ the resolvent of the Laplacian.

Consider, for each $\omega \in \Omega$ fixed, the following approximating equation

$$\begin{cases} dY_\varepsilon(t) - \Delta \bar{\Psi}_\varepsilon(Y_\varepsilon(t) + \sqrt{Q}W(t)) dt = 0, & \text{in } (0, T) \times \mathcal{O}, \\ \bar{\Psi}_\varepsilon(Y_\varepsilon(t) + \sqrt{Q}W(t)) = 0, & \text{on } (0, T) \times \partial\mathcal{O}, \\ Y_\varepsilon(0) = x, & \text{in } \mathcal{O}. \end{cases} \quad (19)$$

where $\bar{\Psi}_\varepsilon(x) = \Psi_\varepsilon(x) + \varepsilon x$, for all $x \in \mathbb{R}$ and Ψ_ε is the Yosida approximation of Ψ , for every $\varepsilon > 0$. By classical existence theory, equation (19) has a unique solution

$$Y_\varepsilon \in C([0, T]; L^2(\mathcal{O})) \cap L^2(0, T; H_0^1(\mathcal{O})),$$

with $Y'_\varepsilon \in L^2(0, T; H^{-1}(\mathcal{O}))$.

For $\mu > 0$ fixed we consider the resolvent of the Laplacian $J_\mu : L^2(\mathcal{O}) \rightarrow L^2(\mathcal{O})$, $J_\mu(x) = (1 - \mu\Delta)^{-1}(x)$, for all $x \in L^2(\mathcal{O})$.

Since J_μ is differentiable we may denote by DJ_μ the Gateaux differential and we see that

$$\frac{d}{dt} J_\mu(Y_\varepsilon(t)) = DJ_\mu(Y_\varepsilon(t)) \frac{\partial}{\partial t} Y_\varepsilon(t) = J_\mu\left(\frac{\partial}{\partial t} Y_\varepsilon(t)\right) \quad (20)$$

for all $t \in [0, T]$.

Now we can prove that $Z = J_\mu Y_\varepsilon$ satisfies *i*), ..., *iii*) from Definition 1.

We can easily see that $J_\mu Y_\varepsilon \in C([0, T]; L^2(\mathcal{O}))$ and then *i*) is satisfied.

Concerning *ii*) we know that, for every $\varepsilon > 0$ fixed, we have

$$\int_0^t \left| J_\mu \frac{\partial}{\partial s} Y_\varepsilon(s) \right|_{-1}^2 ds \leq \int_0^t \left| \frac{\partial}{\partial s} Y_\varepsilon(s) \right|_{-1}^2 ds$$

and

$$\frac{\partial}{\partial s} Y_\varepsilon \in L^2(0, T; H^{-1}(\mathcal{O})).$$

Hence, by (20), we get that

$$\frac{\partial}{\partial t} Z = \frac{d}{dt} J_\mu(Y_\varepsilon) \in L^2(0, T; H^{-1}(\mathcal{O})).$$

Property *iii*) is a direct consequence of $0 \leq g(x) \leq x^2$, for all $x \in (-1, \infty)$. Indeed, we have for every ε and μ fixed that

$$\int_0^t \int_{\mathcal{O}} g(J_\mu Y_\varepsilon(s) + \sqrt{Q}W) d\xi ds \leq \int_0^t \int_{\mathcal{O}} |J_\mu Y_\varepsilon(s) + \sqrt{Q}W|^2 d\xi ds \leq \infty$$

because $J_\mu Y_\varepsilon \in C([0, T]; L^2(\mathcal{O}))$ and $\sqrt{Q}W \in C([0, T] \times \overline{\mathcal{O}})$.

Since $J_\mu Y_\varepsilon$ satisfies i), ii) and iii) we can write Definition 1 for the solution \bar{X} with $Z = J_\mu Y_\varepsilon$, i.e.,

$$\begin{aligned} & \frac{1}{2} \left| \bar{X}(t) - \sqrt{Q}W(t) - J_\mu Y_\varepsilon(t) \right|_{-1}^2 + \int_0^t \int_{\mathcal{O}} g(\bar{X}(s)) d\xi ds \\ & + \int_0^t \int_{\mathcal{O}} (-\Delta)^{-1} \frac{d}{ds} J_\mu Y_\varepsilon(s) \left(\bar{X}(s) - \sqrt{Q}W(s) - J_\mu Y_\varepsilon(s) \right) d\xi ds \\ & \leq \frac{1}{2} |x - J_\mu Y_\varepsilon(0)|_{-1}^2 + \int_0^t \int_{\mathcal{O}} g(J_\mu Y_\varepsilon(s) + \sqrt{Q}W(s)) d\xi ds, \quad \mathbb{P}\text{-a.s.} \end{aligned} \quad (21)$$

Applying J_μ (which is linear) to (19) we get that

$$\frac{d}{dt} [J_\mu Y_\varepsilon(t)] + J_\mu(-\Delta) \left(\bar{\Psi}_\varepsilon(Y_\varepsilon(t) + \sqrt{Q}W(t)) \right) = 0.$$

By Proposition VII 2, $a_1)$ and $a_2)$ from [10] we can rewrite this equation as follows

$$\frac{d}{dt} [J_\mu Y_\varepsilon(t)] - \Delta \left(J_\mu \bar{\Psi}_\varepsilon(Y_\varepsilon(t) + \sqrt{Q}W(t)) \right) = 0. \quad (22)$$

Using (22) in the third term of the left-hand side of (21) we can rewrite it

$$\begin{aligned} & \int_0^t \int_{\mathcal{O}} (-\Delta)^{-1} \frac{d}{ds} J_\mu Y_\varepsilon(s) \left(\bar{X}(s) - \sqrt{Q}W(s) - J_\mu Y_\varepsilon(s) \right) d\xi ds \\ & = \int_0^t \int_{\mathcal{O}} J_\mu \bar{\Psi}_\varepsilon(Y_\varepsilon + \sqrt{Q}W) \left(J_\mu Y_\varepsilon(s) - \bar{X}(s) + \sqrt{Q}W(s) \right) d\xi ds \\ & = \int_0^t \int_{\mathcal{O}} \bar{\Psi}_\varepsilon(Y_\varepsilon + \sqrt{Q}W) \left(Y_\varepsilon + \sqrt{Q}W - J_\mu(\bar{Y} + \sqrt{Q}W) \right) d\xi ds \\ & \quad + \int_0^t \int_{\mathcal{O}} \bar{\Psi}_\varepsilon(Y_\varepsilon + \sqrt{Q}W) \left(J_\mu \sqrt{Q}W - \sqrt{Q}W \right) d\xi ds \\ & \quad + \int_0^t \int_{\mathcal{O}} \bar{\Psi}_\varepsilon(Y_\varepsilon + \sqrt{Q}W) \left((1 - \mu\Delta)^{-2} Y_\varepsilon - Y_\varepsilon \right) d\xi ds \\ & = T_1 + T_2 + T_3, \end{aligned}$$

where $\bar{Y} = \bar{X} - \sqrt{Q}W$.

Since $\overline{\Psi}_\varepsilon(x) = \Psi_\varepsilon(x) + \varepsilon x$, where Ψ_ε is the Yosida approximation of Ψ , and $\overline{g}_\varepsilon(x) = g_\varepsilon(x) + \varepsilon \frac{x^2}{2}$, where g_ε is the Moreau-Yosida approximation of g , we have that $\overline{g}'_\varepsilon = \overline{\Psi}_\varepsilon$ and then, by the definition of the subdifferential, we get that

$$\overline{\Psi}_\varepsilon(x)(x - y) \geq \overline{g}_\varepsilon(x) - \overline{g}_\varepsilon(y).$$

This leads to

$$\begin{aligned} T_1 &= \int_0^t \int_{\mathcal{O}} \overline{\Psi}_\varepsilon(Y_\varepsilon + \sqrt{Q}W) (Y_\varepsilon + \sqrt{Q}W - J_\mu(\overline{Y} + \sqrt{Q}W)) d\xi ds \\ &\geq \int_0^t \int_{\mathcal{O}} (\overline{g}_\varepsilon(Y_\varepsilon + \sqrt{Q}W) - \overline{g}_\varepsilon(J_\mu(\overline{Y} + \sqrt{Q}W))) d\xi ds. \end{aligned}$$

We also have

$$\begin{aligned} -T_2 &= \int_0^t \int_{\mathcal{O}} \overline{\Psi}_\varepsilon(Y_\varepsilon + \sqrt{Q}W) (\sqrt{Q}W - J_\mu\sqrt{Q}W) d\xi ds \\ &\leq \int_0^t \left| \overline{\Psi}_\varepsilon(Y_\varepsilon + \sqrt{Q}W) \right|_{-1} \left| \sqrt{Q}W - J_\mu\sqrt{Q}W \right|_{H_0^1(\mathcal{O})} ds \\ &\leq \left| \overline{\Psi}_\varepsilon(Y_\varepsilon + \sqrt{Q}W) \right|_{L^1((0,T) \times \mathcal{O})} \left| \sqrt{Q}W - J_\mu\sqrt{Q}W \right|_{L^\infty(0,T;H_0^1(\mathcal{O}))} \\ &\leq C \left| \sqrt{Q}W - J_\mu\sqrt{Q}W \right|_{L^\infty(0,T;H_0^1(\mathcal{O}))}, \end{aligned}$$

using Lemma 3 and the fact that $L^1(\mathcal{O}) \subset H^{-1}(\mathcal{O})$, for $\mathcal{O} \in \mathbb{R}$. From (19) follows that

$$\begin{aligned} T_3 &= \int_0^t \int_{\mathcal{O}} (-\Delta)^{-1} \frac{\partial}{\partial s} Y_\varepsilon(s) (Y_\varepsilon(s) - (1 - \mu\Delta)^{-2} Y_\varepsilon(s)) d\xi ds \\ &= \int_0^t \left\langle \frac{\partial}{\partial s} Y_\varepsilon(s), Y_\varepsilon(s) \right\rangle_{-1} ds - \int_0^t \left\langle \frac{d}{ds} J_\mu Y_\varepsilon(s), J_\mu Y_\varepsilon(s) \right\rangle_{-1} ds \\ &= \int_0^t \frac{\partial}{\partial s} \frac{1}{2} |Y_\varepsilon(s)|_{-1}^2 ds - \int_0^t \frac{\partial}{\partial s} \frac{1}{2} |J_\mu Y_\varepsilon(s)|_{-1}^2 ds \\ &= \frac{1}{2} (|Y_\varepsilon(t)|_{-1}^2 - |J_\mu Y_\varepsilon(t)|_{-1}^2) - \frac{1}{2} (|Y_\varepsilon(0)|_{-1}^2 - |J_\mu Y_\varepsilon(0)|_{-1}^2) \\ &\geq -\frac{1}{2} (|Y_\varepsilon(0)|_{-1}^2 - |J_\mu Y_\varepsilon(0)|_{-1}^2) = -\frac{1}{2} (|x|_{-1}^2 - |J_\mu x|_{-1}^2) \end{aligned}$$

since $|J_\mu Y_\varepsilon(t)|_{-1}^2 \leq |Y_\varepsilon(t)|_{-1}^2$ for all $t \in [0, T]$ and x is the starting point of the problem.

Going back to (21) we get that

$$\begin{aligned}
& \frac{1}{2} |\overline{Y}(t) - J_\mu Y_\varepsilon(t)|_{-1}^2 + \int_0^t \int_{\mathcal{O}} g(\overline{X}) d\xi ds + \int_0^t \int_{\mathcal{O}} \overline{g}_\varepsilon(X_\varepsilon) d\xi ds \\
& \leq \frac{1}{2} |x - J_\mu Y_\varepsilon(0)|_{-1}^2 + \int_0^t \int_{\mathcal{O}} g(J_\mu \overline{X}) d\xi ds + \frac{\varepsilon}{2} \int_0^t \int_{\mathcal{O}} (J_\mu \overline{X})^2 d\xi ds \\
& \quad + \int_0^t \int_{\mathcal{O}} g(J_\mu Y_\varepsilon + \sqrt{Q}W) d\xi ds \\
& \quad + C \left| \sqrt{Q}W - J_\mu \sqrt{Q}W \right|_{L^\infty(0,T;H_0^1(\mathcal{O}))} \\
& \quad + \frac{1}{2} \left(|x|_{-1}^2 - |J_\mu x|_{-1}^2 \right), \quad \mathbb{P}\text{-a.s.}
\end{aligned}
\tag{23}$$

We firstly pass to the *liminf* for $\varepsilon \rightarrow 0$, with $\mu > 0$ fixed, as follows. Arguing as we did in the proof of existence we have that

$$\liminf_{\varepsilon \rightarrow 0} \int_0^t \int_{\mathcal{O}} \overline{g}_\varepsilon(X_\varepsilon(s)) d\xi ds \geq \int_0^t \int_{\mathcal{O}} g(X(s)) d\xi ds, \quad \text{for all } t \in [0, T].$$

We shall now pass to the *liminf* for $\varepsilon \rightarrow 0$, with $\mu > 0$ fixed, in

$$\int_0^t \int_{\mathcal{O}} g(J_\mu Y_\varepsilon(s) + \sqrt{Q}W(s)) d\xi ds.$$

We know by (7) that $\{Y_\varepsilon\}_\varepsilon$ is bounded in $C([0, T]; L^2(\mathcal{O}))$ and considering (15) we get that

$$Y_\varepsilon(t) \rightarrow Y(t), \quad \text{weakly in } L^2(\mathcal{O}), \quad \text{for all } t \in [0, T],$$

as $\varepsilon \rightarrow 0$. On the other hand, we know that, for every $\mu > 0$ fixed, we have that $J_\mu : L^2(\mathcal{O}) \rightarrow L^2(\mathcal{O})$ is compact and then

$$J_\mu Y_\varepsilon(t) \rightarrow J_\mu Y(t), \quad \text{strongly in } L^2(\mathcal{O}), \quad \text{for all } t \in [0, T],$$

as $\varepsilon \rightarrow 0$.

Since $g : \mathbb{R} \rightarrow \mathbb{R}$ is continuous and $\sqrt{Q}W \in C([0, T] \times \overline{\mathcal{O}})$ we have that

$$g(J_\mu Y_\varepsilon + \sqrt{Q}W) \rightarrow g(J_\mu Y + \sqrt{Q}W), \quad \text{a.e. on } [0, T] \times \mathcal{O}. \tag{24}$$

We also know that $J_\mu : L^2(\mathcal{O}) \rightarrow C(\overline{\mathcal{O}})$ is continuous and then $\{J_\mu Y_\varepsilon + \sqrt{Q}W\}_\varepsilon$ is bounded in $C([0, T] \times \overline{\mathcal{O}})$. We obtain that

$$g(J_\mu Y_\varepsilon + \sqrt{Q}W) \leq |J_\mu Y_\varepsilon + \sqrt{Q}W|^2 \leq C, \quad \text{a.e. on } [0, T] \times \mathcal{O}. \tag{25}$$

Consequently, by Lebesgue's dominated convergence theorem, we get from (24) and (25) that

$$\lim_{\varepsilon \rightarrow 0} \int_0^t \int_{\mathcal{O}} g(J_\mu Y_\varepsilon + \sqrt{Q}W) d\xi ds = \int_0^t \int_{\mathcal{O}} g(J_\mu Y + \sqrt{Q}W) d\xi ds.$$

Going back to (23) and passing to the *liminf* for $\varepsilon \rightarrow 0$, with $\mu > 0$ fixed, we get that

$$\begin{aligned}
& \frac{1}{2} |\overline{Y}(t) - J_\mu Y(t)|_{-1}^2 + \int_0^t \int_{\mathcal{O}} g(X) d\xi ds + \int_0^t \int_{\mathcal{O}} g(\overline{X}) d\xi ds \quad (26) \\
\leq & \frac{1}{2} |x - J_\mu x|_{-1}^2 + \int_0^t \int_{\mathcal{O}} g(J_\mu \overline{X}) d\xi ds + \int_0^t \int_{\mathcal{O}} g(J_\mu Y + \sqrt{Q}W) d\xi ds \\
& + C \left| \sqrt{Q}W - J_\mu \sqrt{Q}W \right|_{L^\infty(0,T;H_0^1(\mathcal{O}))} \\
& + \frac{1}{2} \left(|x|_{-1}^2 - |J_\mu x|_{-1}^2 \right), \quad \mathbb{P}\text{-a.s.}
\end{aligned}$$

We shall now pass to the *liminf* for $\mu \rightarrow 0$.

Firstly we discuss the following term

$$\int_0^t \int_{\mathcal{O}} [g(J_\mu \overline{X}(s)) - g(\overline{X}(s))] d\xi ds.$$

Since

$$J_\mu \overline{X}(s) \rightarrow \overline{X}(s), \text{ strongly in } L^2(\mathcal{O}), \text{ for all } s \in [0, T]$$

as $\mu \rightarrow 0$ and since $g : \mathbb{R} \rightarrow \mathbb{R}$ is continuous, we have that

$$g(J_\mu \overline{X}(s)) \rightarrow g(\overline{X}(s)), \text{ a.e. on } \mathcal{O}, \text{ for all } s \in [0, T]$$

as $\mu \rightarrow 0$.

By the Egorov theorem we get that, for every $\delta > 0$, there exists a set E_δ with the Lebesgue's measure $|E_\delta| < \delta$, such that

$$g(J_\mu \overline{X}) - g(\overline{X}) \rightarrow 0, \text{ uniformly on } \mathcal{O} \setminus E_\delta$$

as $\mu \rightarrow 0$, for all $t \in [0, T]$ and then

$$\lim_{\mu \rightarrow 0} \int_{\mathcal{O}} [g(J_\mu \overline{X}) - g(\overline{X})] d\xi = \lim_{\mu \rightarrow 0} \int_{E_\delta} [g(J_\mu \overline{X}) - g(\overline{X})] d\xi$$

for all $t \in [0, T]$.

On the other hand we have

$$\begin{aligned}
& \int_{E_\delta} [g(J_\mu \overline{X}) - g(\overline{X})] d\xi \\
\leq & \left(\int_{E_\delta} 1 d\xi \right)^{1/2} \left(\int_{E_\delta} [g(J_\mu \overline{X}) - g(\overline{X})]^2 d\xi \right)^{1/2} \\
\leq & 2 |E_\delta|^{1/2} \left(\int_{E_\delta} \overline{X}^4 d\xi \right)^{1/2} \leq C\delta, \text{ a.e. on } [0, T].
\end{aligned}$$

(Indeed, we can easily see that $|J_\mu \overline{X}|_{L^4(E_\delta)} \leq |\overline{X}|_{L^4(E_\delta)}$ by using the same argument as in [4] to obtain 3.25).

Then, we get that

$$\lim_{\mu \rightarrow 0} \int_{\mathcal{O}} [g(J_{\mu} \overline{X}) - g(\overline{X})] d\xi = 0, \text{ a.e. on } [0, T].$$

We also know from (6) that

$$\begin{aligned} \int_{\mathcal{O}} [g(J_{\mu} \overline{X}(t)) - g(\overline{X}(t))] d\xi &\leq 2 \int_{\mathcal{O}} |\overline{X}(t)|^2 d\xi \\ &\leq 2 \sup_{t \in [0, T]} \int_{\mathcal{O}} |\overline{X}(t)|^2 d\xi \leq C, \text{ a.e. on } [0, T]. \end{aligned}$$

Using Lebesgue's dominated convergence theorem we get that

$$\lim_{\mu \rightarrow 0} \int_0^t \int_{\mathcal{O}} [g(J_{\mu} \overline{X}(s)) - g(\overline{X}(s))] d\xi ds = 0.$$

Using the same argument we get that

$$\lim_{\mu \rightarrow 0} \int_0^t \int_{\mathcal{O}} [g(J_{\mu} Y(s) + \sqrt{Q}W(s)) - g(X(s))] d\xi ds = 0.$$

In order to conclude the proof we only need to mention that, for each $\omega \in \Omega$ fixed, we have

$$\lim_{\mu \rightarrow 0} \left| \sqrt{Q}W - J_{\mu} \sqrt{Q}W \right|_{L^{\infty}(0, T; H_0^1(\mathcal{O}))} = 0, \quad (27)$$

which is a consequence of the fact that $\sqrt{Q}W \in L^{\infty}(0, T; H_0^1(\mathcal{O}))$

Going back to (26) we can pass to the *liminf* for $\mu \rightarrow 0$ and get that, for each $\omega \in \Omega$ fixed, we have

$$|\overline{X}(t) - X(t)|_{-1} = 0,$$

for all $t \in [0, T]$, and that assure the uniqueness of the solution. ■

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